THEOREMS ON TOPOLOGICAL SEMIGROUPS

By

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CHAPTER I

PRELIMINARIES

A <u>topological semigroup</u> is a function $m: S \times S \longrightarrow S$ such that:

(i) S is a non-void Hausdorff space;

(ii) m is a continuous function;

(iii) m is associative. That is, for any x, y, z in S we have $m(x,m(y,z)) = m(m(x,y),z)$.

In general, when the above is satisfied, we will say that "S is a semigroup," and denote simply xy or $x \cdot y$ for $m(x,y)$. The semigroups for which the underlying spaces are discrete are called <u>discrete semigroups</u>, or <u>algebraic semigroups</u>. <u>In all that follows let S be always a semigroup.</u> An element u (z) is a <u>unit</u> (<u>zero</u>) of S if and only if $xu = x = ux$ ($xz=z=zx$) for all x in S. If S has a unit (zero), then S has a unique unit (zero). A <u>left</u> (<u>right</u>, <u>two-sided</u>) <u>ideal</u> of S is a non-void subset T of S such that $ST \subset T$ ($TS \subset T, ST \cup TS \subset T$). A two-sided ideal of S will, also, be called an <u>ideal</u> of S. We shall denote by K, or $K(S)$, the (unique if it exists) <u>minimal ideal</u> of S. An idempotent of S is an element e of S such that $e^2 = ee = e$. If $T \subset S$, then $E(T)$ will be the set of all idempotents in

T; we also write E for E(S) if there is no ambiguity. If e is an idempotent of S, let H(e) be the set $eSe \cap \{x: x \in eS \cap Se\}$ which is seen to be the maximal subgroup of S having e as its unit.

In general, if X is a linear space then X^* will be the dual (= conjugate) space of X. We write A^- for the topological closure of A, and $\square$ for the empty set. The symbol $A \setminus B$ is used for the complement of B in A.

We are now in position to state some well-known theorems which will be used in the sequel.

<u>Theorem 1</u>.[29]. If S is compact, then S has minimal ideals of all three categories (left, right and two-sided). Moreover, each such is closed; and if S is connected then each of these is connected. A minimal (two-sided) ideal is unique whenever it exists.

In general, proofs for well-known theorems are omitted and the references are given. Only the theorems that are supposed to be new, and non-trivial, are proved.

<u>Theorem 2</u> [29]. If S has a minimal left ideal and a minimal right ideal (for instance, if S is compact), then the minimal ideal K exists and is the union of all minimal left (right) ideals, and, also, the union of maximal subgroups, each of these being eSe for some e in $E \cap K$; and conversely.

A useful consequence of this is the following

Corollary. If S is compact then $E \neq \square$.

Theorem 3 [27],[28]. Any subgroup of S is contained in a maximal subgroup; no two maximal subgroups intersect. If S is compact then each maximal subgroup is closed, and is a topological group.

We say that S is normal if $xS = Sx$ for every x in S. Clearly, all commutative semigroups are normal.

Theorem 4. If S is compact and normal and if T is a dense subset of S, then $K = \cap\{tS: t \in T\}^-$ and is a topological group.

Proof: A proof of this for the case in which S is commutative is contained in [11]. The proof applies equally well to this case.

We define, for any x in S,

$$\Gamma_n(x) = \{x^n, x^{n+1}, \ldots\}^- ,$$

$$\Gamma(x) = \Gamma_1(x), \quad \text{and}$$

$$N(x) = \cap\{\Gamma_n(x): n \geq 1\} .$$

Theorem 5 (Koch[15], Numakura [19]). If x is an element in S such that $\Gamma(x)$ is compact, then $N(x)$ is the minimal ideal of $\Gamma(x)$ as well as the unique maximal subgroup of $\Gamma(x)$. (See also [16]).

A <u>quasi-ordering</u> on a set X is a reflexive and transitive relation on X. A <u>partial-ordering</u> is an anti-symmetric quasi-ordering.

<u>Theorem 6</u> [30]. Let $(X,\leq)$ be a non-void topological space endowed with a quasi-ordering $\leq$ such that for each t in X the set $\{x: x \in X, x \leq t\}$ is closed in X. Then $(X,\leq)$ has a minimal element.

Let X be a Banach space [10] over the field of complex numbers, let X* be the dual space of X; that is, X* is the set of all bounded linear functionals on X which with the obvious addition, multiplication, scalar multiplication and the operator norm becomes a Banach space [10]. Several topologizations on X* are known, the one which is very useful to us is the weak-star topology. The letter C will be the set of complex numbers.

<u>Definition 1</u>. Let X be a Banach space over the complex field C and let X* be its dual space. Define, for each x in X, $\tilde{x}: X^* \longrightarrow C$ by $\tilde{x}(T) = T(x)$ for all T in X*. Then the <u>weak-star topology</u> for X* is the weakest (=smallest) topology for X* such that $\tilde{x}$ is continuous for every x in X.

<u>Theorem 7</u> (Alaoglu [1]). The closed unit ball in the dual space X* of the Banach-space X is compact in the weak-star topology.

Let X be a locally compact Hausdorff space and let B(X) be the Borel ring of X, i.e., the σ-ring generated by the compact subsets of X. By a Borel measure on X we mean a countably additive complex-valued set function defined on B(X).

Definition 2. Let μ be a Borel measure on a locally compact Hausdorff space X. Then the total variation of μ, denoted by $||\mu||$, is defined as

$$||\mu|| = \sup \sum_{i=1}^{n} |\mu(E_i)| ,$$

where the supremum is taken over all finite disjoint sequences $\{E_i : E_i \in B(X), i = 1,\ldots,n\}$.

Definition 3. Let X be a locally compact Hausdorff space. A Borel measure μ on X is said to be regular if for each E in B(X) and $\varepsilon > 0$ there is a compact set F contained in E and an open set G containing E such that $|\mu(A)| < \varepsilon$ for every A in B(X) with $A \subset G \setminus F$.

For any locally compact Hausdorff space X, define M(X) to be the set of all (complex-valued) regular Borel measures on X for which $||\mu||$ is finite. We remark that M(X) is a Banach-space.

Theorem 8 (Riez-Kakutani representation theorem [13]). Let X be a compact Hausdorff space and let C(X)

be the Banach space of all complex-valued continuous functions on X. Then M(X) is isometrically isomorphic to C(X)* such that the corresponding elements T and μ satisfy the identity

$$T(f) = \int_X f(x)\, \mu(dx), \qquad f \in C(X).$$

In constructing convolution semigroups of measures (Chapter IV), theorems 7 and 8 are extremely vital to us. We also note that Theorem 8 remains true for a locally compact X, in which case C(X) should be replaced by $C_o(X)$, the complex-valued continuous functions vanishing at infinity.

CHAPTER II

SUMMARY OF RESULTS

This dissertation consists mainly of three topics on topological semigroups, viz., generalized character semigroups; not necessarily convolution semigroups of measures; convolution semigroups of modular functions.

In all that follows, let S be a topological semigroup.

An elegant structure theorem developed by Š. Schwarz states that if S is a finite abelian or compact abelian semigroup admitting relative inverses, then the character semigroup $S^{\wedge}$ of S may be decomposed into a disjoint union of character groups of certain maximal subgroups of S. We generalize, among other things, this theorem to a broader class of semigroups, the so called "pseudo-invertible semigroups" which contains the finite semigroups, the semigroups admitting relative inverses and many others as subclasses. We, also, relax the range of the characters from the semigroup of complex numbers to a certain general semigroup.

Let S be compact, and let $\tilde{S}$ be the set of normalized non-negative regular Borel measures on S. Under

convolution of measures and with the topology induced by the weak-star topology for the dual $C(S)^*$ of the Banach-space of all complex-valued continuous functions on S, $\tilde{S}$ forms a compact topological semigroup. J. G. Wendel in 1954 first introduced the convolution semigroup $\tilde{S}$ of measures for the case in which S is a compact topological group; in 1959, I. Glicksberg was then able to generalize this to the case in which S is a compact abelian semigroup. He determined the closed subgroups of $\tilde{S}$ when S is abelian. In his proof of a fundamental theorem: the carrier of the kernel is the kernel of the carrier, Glicksberg made use of the fact that the kernel of a compact abelian semigroup is a topological group. In the not necessarily abelian case, this is no longer true. However, we are still able to prove the fundamental theorem using techniques developed by A. D. Wallace. We also determine, among other things, the carriers of the maximal subgroups of $\tilde{S}$, for the class of S in which every compact simple subsemigroup is a group, they are just the normalizers of the closed subgroups of S each with respect to the corresponding maximal subgroup.

Professor A. D. Wallace asked if the process of forming the convolution semigroup of measures could be generalized to a more general class of set functions; namely, modular functions. We settle this question in the affirmative under the slight restriction that modular

functions are isotone. Indeed, we prove that every normalized regular isotonic modular function may be extended uniquely to a regular Borel measure without increasing the norm. Thus, if $S^{\#}$ denotes the collection of all normalized regular isotonic modular functions on S, then $S^{\#}$ is essentially the same as the convolution semigroup $\tilde{S}$ of S.

CHAPTER III

GENERALIZED CHARACTER SEMIGROUPS

An elegant structure theorem due to Š. Schwarz [23],[24] asserts that if S is a finite abelian or a compact abelian semigroup admitting relative inverses (see Definition 7), then the character semigroup of S may be decomposed into a disjoint union of character groups of certain maximal subgroups of S. In the following, among other things, we generalize this Schwarz theorem to a broader class of semigroups, the so called "pseudo-invertible semigroups." We, also, relax the range of the characters from the semigroup of complex numbers to a certain general semigroup.

Throughout this chapter, let S be always a compact commutative semigroup, unless otherwise stated. By a character of S is meant a continuous homomorphism of S into the multiplicative semigroup C of the complex numbers endowed with the usual Euclidean topology. The collection of all characters of S, with the value-wise multiplication of functions, endowed with the compact-open topology, forms a semigroup which will be denoted by $(S,C)^{\wedge}$ or simply $S^{\wedge}$,

and will be called the <u>character semigroup</u> of S. Hewitt and Zuckerman [12] use the term <u>semicharacter</u>, in the discrete case, for not identically zero characters. Here we use $(\hat{S},C)$ or simply $\hat{S}$, as distinguished from $S^\wedge$, to denote the collection of semicharacters of S. We note that $\hat{S}$, in general, need not be a semigroup. We first draw attention to the fact that if χ is a character of S, then $|\chi(x)| \leq 1$ for every x in S. For, otherwise $\chi(S)$ would not be compact. Thus, in the study of characters, only the unit disc $\{z: |z| \leq 1\}$ of the complex numbers is used. Let us write D for this unit disc. The set D itself forms an important semigroup which is compact, connected, commutative, cancellable,[1] has zero 0 and unit 1; moreover the circumference $\{z: |z| = 1\}$ of D is the maximal subgroup H(1) and $D\backslash H(1)$ is an ideal. However, only some of these are needed as we shall see below.

Throughout the rest of this chapter, let T be an arbitrary, but fixed, compact commutative cancellable semigroup with zero z and unit u[2] such that $T\backslash H(u)$ is a subsemigroup of T. By a <u>generalized character</u> of S is meant a continuous homomorphism of S to T. As in the

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[1]A semigroup S is <u>cancellable</u> if and only if for any non-zero elements a, b, c in S such that ab=ac or ba=ca, then b=c.

[2]It is to be understood that $z \neq u$.

case of character semigroup, the collection $(S,T)^\wedge$ of all generalized characters of S, with value-wise multiplication of mappings and the compact-open topology, forms a commutative semigroup which will be called the <u>generalized character semigroup</u> of S. We write $(\hat{S},T)$ for the collection of all not identically zero elements in $(S,T)^\wedge$. It is quite easy to see that if S is a group, then $(\hat{S},T) = (\hat{S},H(u))$ and $(\hat{S},T)$ is a group.

<u>Theorem 9</u>. If S is discrete, then $(S,T)^\wedge$ is compact.

<u>Proof</u>. Since S is discrete, the compact-open topology on $(S,T)^\wedge$ is the relative topology, on the set (not topologized) $(S,T)^\wedge$, of the Tychonoff product topology on the product $P\{T: s \in S\}$ of S copies of the compact space T. $P\{T: s \in S\}$ is compact by the Tychonoff theorem. The compactness of $(S,T)^\wedge$ now follows from the fact that $(S,T)^\wedge$ is a closed subset of $P\{T: s \in S\}$.

<u>Definition 4</u>. For any χ in $(S,T)^\wedge$, the <u>support</u> of χ, $sp(\chi)$, is the set $\{x: x \in S, \chi(x) \neq z\}$.

We have immediately $sp(\chi_1 \cdot \chi_2) = sp(\chi_1) \cap sp(\chi_2)$ for any χ_1, χ_2 in $(S,T)^\wedge$. It is clear that if χ in $(S,T)^\wedge$ is not identically zero, $sp(\chi)$ is an open subsemigroup of S; such an open subsemigroup will be called a <u>supporting</u>

subsemigroup. Since the support of the zero generalized character is the void set $\square$, as a convenience we also call $\square$ a supporting subsemigroup. We write henceforth, $\mathcal{P}(S)$ or $\mathcal{P}$ for the collection of all supporting subsemigroups of S.

Definition 5 [22]. The Rees partial-ordering $\leq$ on the set E of idempotents in S is the subset $\{(e_1,e_2): (e_1,e_2) \in E \times E,\ e_1e_2 = e_1\}$. If (e_1,e_2) is in $\leq$ we write, equivalently, $e_1 \leq e_2$.

Lemma 1. If S_o is a compact subsemigroup of S, then $E(S_o)$ has a unique minimal element with respect to the Rees partial-ordering $\leq$.

Proof. Since $E(S_o)$ is always a closed subset of S_o and S_o is compact, $E(S_o)$ is compact. By the corollary to Theorem 2, $E(S_o) \neq \square$, it now follows from Theorem 6 that $(E(S_o), \leq)$ has a minimal element, say e_o. If there were another minimal element e_1 in $E(S_o)$, then one obtains $e_1 = e_oe_1 = e_o$. This proves the uniqueness.

Definition 6. If $P \in \mathcal{P}(S)$, then $\sigma(P)$ is the set $\{\chi: \chi \in (S,T)^\wedge,\ sp(\chi) = P\}$.

For each P in $\mathcal{P}(S)$, $\sigma(P)$ is easily seen to be a cancellable subsemigroup of $(S,T)^\wedge$. In general, $(S,T)^\wedge$

may be decomposed into the union of the disjoint family $\{\sigma(P): P \in \mathcal{P}(S)\}$ of cancellable subsemigroups of $(S,T)^\wedge$.

Definition 7 [3]. A semigroup S admitting relative inverses is a semigroup such that to each x in S there is a pair (e,x') in $E \times S$ such that $xe = x = ex$ and $xx' = e = x'x$.

A well-known result of A. H. Clifford [3] says that a semigroup is a semigroup admitting relative inverses if and only if it is the disjoint union of its maximal subgroups. The more general class of semigroups that we are interested in is the following.

Definition 8 [9]. A semigroup S is pseudo-invertible if and only if, to each element x in S there is an $\bar{x}$ in S such that (i) $x\bar{x} = \bar{x}x$; (ii) $\bar{x}x^{n+1} = x^n$ for some positive integer n, and (iii) $\bar{x}^2x = \bar{x}$.

The element $\bar{x}$ satisfying conditions (i), (ii) and (iii) above turns out to be unique if it exists [9], in which case it is called the pseudo-inverse of x. A semigroup S is pseudo-invertible if and only if, to every x in S there is an integer $n > 0$ such that x^n is in some subgroup of S [9], [17], [18]. From this, one sees that the class of pseudo-invertible semigroups includes all semigroups admitting relative inverses, all periodic

semigroups; all semigroups of matrices; all finite dimensional affine semigroups (for definition of an affine semigroup, see [4]) and many others.

<u>Lemma 2</u>. Let S be a compact commutative pseudo-invertible semigroup. Then each supporting subsemigroup P of S is open and closed. Therefore, if $P \neq \square$ then P has a unique minimal idempotent e_P with respec to the Rees partial-ordering on E(P).

<u>Proof</u>: To show each P in $\mathcal{P}$ is open and closed, we consider only $P \neq \square$; for $P = \square$ is a trivial case. For any non-void P in $\mathcal{P}$, there is a χ in $(S,T)^\wedge$ such that $P = \{x: x \in S, \chi(x) \neq z\}$. We show, for pseudo-invertible S, also, $P = \{x: x \in S, \chi(x) \in H(u)\}$ and consequently P is open and closed. To this end, let x be an arbitrary element of the non-void set $P = \{x: x \in S, \chi(x) \neq z\}$. Then since S is pseudo-invertible, there is a positive integer n such that $x^n \in H(e)$ for some e in E; thus, since $E(T) = \{z,u\}$, we must have $\chi(x)^n = \chi(x^n) \in H(u)$. Consequently, since $T\backslash H(u)$ is a subsemigroup, we obtain $\chi(x) \in H(u)$. This proves $\{x: x \in S, \chi(x) \neq z\} = P = \{x: x \in S, \chi(x) \in H(u)\}$, so that P is open and closed.

The set P being a closed subsemigroup of the compact S, by Lemma 1, E(P) has a unique minimal element e_P with respect to the Rees partial ordering.

In the following, for $Q \subset S$, e_Q will be the least idempotent in Q if it exists.

<u>Theorem 10</u>. Let S be a compact commutative pseudo-invertible semigroup. Then the generalized character semigroup $(S,T)^\wedge$ of S may be decomposed into the union of the disjoint family $\{(\widehat{H(e_P)},T): P \in \mathcal{P}\}$ of groups, where we agree that $(\hat{H}_{\neg},T) = \{0\}$.

The proof of this theorem is contained in the following two lemmas. It should be noted that when $T = C$, $(\widehat{H(e)}, T)$ is the familier character group $\widehat{H(e)}$.

<u>Lemma 3</u>. Under the hypothesis of Theorem 10, for any non-void P in $\mathcal{P}(S)$, the maximal subgroup $H(e_P)$ of S is the kernel of P; and the mapping $r_P: P \longrightarrow H(e_P)$ which takes every x in P to xe_P is a (continuous) retraction of P onto $H(e_P)$.

<u>Proof</u>. Let x be an arbitrary element of S. Let $\Gamma(x) = \{x^n: n \geq 1\}^-$ and $N(x) = \cap\{x^n\Gamma(x): n \geq 1\}$, then $N(x)$ is the kernel of $\Gamma(x)$ as well as, since S is compact, a closed subgroup of S. Thus $N(x)$ contains a unique idempotent which is designated by e_x. If x is in P we have $e_xe_P = e_P$ and hence the unique idempotent in $N(xe_P)$ is e_P. Therefore, $xe_P = (xe_P)e_P \in N(xe_P) \subset H(e_P)$ for all x in P. To show $H(e_P)$ is an ideal of P, we first show

that $H(e_p) \subset P$. This is true since there is a χ_p in $(S,T)^\wedge$ such that an element x of S is in P if and only if $\chi_p(x) \neq z$; so $\chi_p(e_p) \neq z$ (consequently $\chi_p(e_p) = u$) and thus $\chi_p(x) \neq z$ for all x in $H(e_p)$. Now $P\,H(e_p) = P\,H(e_p)\,e_p \subset P\,e_p \subset H(e_p)$ shows that $H(e_p)$ is an ideal of P; since it is a group it must be the minimal ideal of P. The fact that $r_p\colon P \longrightarrow H(e_p)$ is a continuous retraction onto is then evident.

<u>Lemma 4</u>. Under the hypothesis of Theorem 10, for any non-void P in $\mathcal{P}(S)$, $\sigma(P) = \{\chi\colon \chi \in (S,T)^\wedge,\ sp(\chi) = P\}$ is isomorphic to the generalized character group $(\widehat{H(e_p)},T)$ of the maximal subgroup $H(e_p)$ of S. In particular, if T = C, $\sigma(P)$ is also homeomorphic with the character group $\widehat{H(e_p)}$ of $H(e_p)$.

<u>Proof</u>. Let $h\colon \sigma(P) \longrightarrow (\widehat{H(e_p)},T)$ be the mapping which takes each χ in $\sigma(P)$ to $\chi|H(e_p)$. Clearly $\chi \in \sigma(P)$ implies $\chi|H(e_p) \in (\widehat{H(e_p)},T)$. We have $h(\chi_1\cdot\chi_2) = (\chi_1\cdot\chi_2)|H(e_p) = (\chi_1|H(e_p))\cdot(\chi_2|H(e_p)) = h(\chi_1)\cdot h(\chi_2)$ so that h is a homomorphism. To show h is an isomorphism, we show each φ in $(\widehat{H(e_p)},T)$ may be extended, uniquely, to a χ in $\sigma(P)$. To this end, we define, for φ in $(\widehat{H(e_p)},T)$.

$$\chi = \begin{cases} 0 & \text{on } S\setminus P \\ \varphi\circ r_p & \text{on } P. \end{cases}$$

This is continuous because r_p is continuous and P is open and closed. A routine verification shows that χ is an extension of φ to an element in $\sigma(P)$. Such an extension is unique as we shall now see. If χ' is any element in $\sigma(P)$ with $\chi'|H(e_p) = \varphi$, then $\chi'(x) = z = \chi(x)$ for all x in $S \setminus P$ and

$$\chi'(x) = \chi'(x)\cdot u = \chi'(x)\cdot\chi'(e_p) = \chi'(xe_p) = \varphi(xe_p) =$$

$$\varphi \circ r_p(x) = \chi(x), \quad \text{for all } x \text{ in } P.$$

Therefore, h is an isomorphism of $\sigma(P)$ onto $(\widehat{H(e_p)},T)$. It remains to show, in the case $T = C$, that $h: \sigma(P) \longrightarrow \widehat{H(e_p)}$ is also a homeomorphism. This follows from the fact that h is one-to-one, continuous and that $\widehat{H(e_p)}$ is discrete [20].

<u>Corollary</u>. If S satisfies the hypotheses of Theorem 10, and if S is connected, then $\mathcal{P} = \{S, \sqcap\}$ and hence $(S,T)^\wedge = (\widehat{H(e_s)},T) \cup \{0\} = (H(e_s),T)^\wedge$.

CHAPTER IV

NOT NECESSARILY ABELIAN CONVOLUTION SEMIGROUPS OF MEASURES

4.1. Introduction. Throughout this chapter, let S be always a compact semigroup, and let $\tilde{S}$ be the set of normalized non-negative regular Borel measures on S. Under convolution and with the relative topology induced by the weak-star topology for C(S)*, the dual of the Banach space of all complex-valued continuous functions on S, $\tilde{S}$ forms a compact semigroup. In his pioneer work [31](1954) Wendel introduced the convolution semigroup $\tilde{S}$ of measures, when S is a compact topological group, and he identified the idempotents in $\tilde{S}$ with the Haar measures on the closed subgroups of S.[1] Glicksberg [11] was then able to generalize this to the case in which S is a compact abelian semigroup, and he carried Wendel's results steps further and deeper. He determined, among other things, the closed subgroups of $\tilde{S}$ when S is abelian or a not necessarily abelian group. Recently, Collins has published a series of papers

[1]Professor A. D. Wallace has kindly pointed out that Wendel's result was also proved by Kawada and Ito[14].

[5], [6], [7] related to this. There are, in addition, the papers of Pym [21] and of Collins and Koch [8], and Schwarz [25]. Of these, the author feels that Glicksberg's work is the deepest.

The main purpose of this chapter is to generalize Glicksberg's results to the case in which S is a not necessarily abelian. In his proof of a fundamental theorem: the carrier of the kernel is the kernel of the carrier, Glicksberg [11] made use of the fact that the kernel of a compact abelian semigroup is a topological group. In the not necessarily abelian case, this is no longer true. However, we are still able to prove Glicksberg's theorem due to the fact that the kernel of a compact (not necessarily abelian) semigroup is a union of topological groups (Theorem 2, Chap. I), though the kernel need not be itself a group. As a natural corollary to our generalization of Glicksberg's theorem, we have: the carrier of a simple subsemigroup is simple.[2] This, in a sense, is a satisfactory generalization of the main result of Collins [6] that the carrier of each idempotent measure is a compact simple subsemigroup, since each idempotent itself forms a simple subsemigroup. We prove, among other things, that for any μ in $\tilde{S}$, $(1/N)\Sigma_{n=1}^{N}\mu^{n}$ converges, in the

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[2] A semigroup S is said to be simple if S is its only ideal.

weak-star topology, to an idempotent measure which is carried on the kernel of a certain compact subsemigroup of S. In the case in which E(S) commutes, the kernel becomes a subgroup and the idempotent measure is the Haar measure on this subgroup.

If S is such that the carrier of each idempotent measure is a group (this is true if, for instance, S is normal or the idempotents commute), then the carriers of the maximal subgroups of $\tilde{S}$ are determined: they are just the normalizers[3] of the closed subgroups of S, each in the corresponding maximal subgroup of S. More precisely, if η is an idempotent in $\tilde{S}$, then the carrier of $H(\eta)$ is the normalizer of carrier η in the maximal subgroup of S containing carrier η. Conversely, if g is a closed subgroup of S and if G is the maximal subgroup of S containing g, then the normalizer N(g: G) is the carrier of the maximal subgroup $H(\eta)$ of $\tilde{S}$ such that $\eta^2=\eta$ and carrier η = g.

4.2. Definitions and Lemmas. By virtue of the Riez-Kakutani representation theorem (Theorem 8), $\tilde{S}$ as a subset of M(S) may be identified isometrically with a

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[3]Let g be a subgroup of S, and let H be a subgroup of S containing g, then the normalizer of g in H is the set $\{x: x \in H, xg = gx\}$, which is denoted by N(g: H).

certain subset of $C(S)^*$, and this allows us to use the same symbol for the measures on S as well as the functionals on C(S), writing $\mu(f) = \int_S f(x)\mu(dx)$. The norm $||\mu||$ of μ in $C(S)^*$ is the total variation as a measure on S, and the closed unit ball of $C(S)^*$ is, by the Alaoglu theorem (Theorem 7, Chap. I), compact in the weak-star topology, and hence so is its closed subspace $\tilde{S} = \{\mu: \mu \geq 0, ||\mu|| = 1\}$. If, furthermore, we introduce in $\tilde{S}$ the <u>convolution</u> multiplication:

$$\mu\nu(f) = \int_S \int_S f(xy)\mu(dx)\nu(dy)$$

for every f in C(S), then we have the following

<u>Theorem 11</u>.[11]. $\tilde{S}$ with convolution and weak-star topology forms a compact semigroup. Moreover, $\tilde{S}$ is abelian if S is abelian.

<u>Henceforth $\tilde{S}$ will stand for this semigroup</u>. By the <u>carrier</u> of μ, for μ in $\tilde{S}$, we mean the unique minimal closed set A in S such that $\mu(A) = 1$; such an A exists since it is the complement of the union of all open μ-null sets. For $\Sigma \subset \tilde{S}$, we define carrier Σ to be the closure of the union of all carrier μ, μ running over Σ.

The following theorem is our fundamental tool throughout the sequel. It was proved by Wendel [31] for

the case in which S is a compact group and by Glicksberg [11] for compact semigroup. Pym [21, p.686] remarked that Glicksberg's proof of this depends on the commutativity of S, this however is erroneous. For the sake of convenience, we include Glicksberg's proof here.

<u>Theorem 12</u> [11, p.54]. If μ and ν are in $\tilde{S}$, then

$$\text{carrier } \mu\nu = (\text{carrier } \mu)(\text{carrier } \nu).$$

<u>Proof</u>: Let A and B be the respective carriers of μ and ν. Since each is compact so is $A \cdot B$ which in particular is then a Borel set. Thus by the regularity of $\mu\nu$, for $\epsilon > 0$ we have an open U containing $A \cdot B$ for which $\mu\nu(U) \leq \mu\nu(A \cdot B) + \epsilon$. Since S is normal Urysohn's lemma applies to yield an F in C(S) with $\varphi_{A \cdot B} \leq F \leq \varphi_U$ (where φ_E is the characteristic function of E), i.e., $0 \leq F \leq 1$ and $F = 0$ on $S \setminus U$, $= 1$ on $A \cdot B$. But it is clear that $\varphi_A(x)\varphi_B(y) \leq F(xy)$ for all x, y in S, and thus

$$\begin{aligned} 1 = \mu(A) \cdot \nu(B) &= \iint \varphi_A(x)\varphi_B(y)\mu(dx)\nu(dy) \\ &\leq \iint F(xy)\mu(dx)\nu(dy) \\ &= \int F(x)\mu\nu(dx) \leq \mu\nu(U) \\ &\leq \mu\nu(A \cdot B) + \epsilon \leq 1 + \epsilon . \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\mu\nu(A \cdot B) = 1$. Moreover if U is now an open set with $(A \cdot B) \cap U \neq \square$ then we can find

open sets V and W for which $V \cap A \neq \square$, $W \cap B \neq \square$, and $V^{-}.W^{-} \subset U$; choosing an F in C(S) with $\mu(V^{-}) \geq \mu(V) > 0$, $\nu(W^{-}) \geq \nu(W) > 0$ to show $\mu\nu(U) > 0$. Hence $A \cdot B$ is indeed the carrier of $\mu\nu$.

As a consequence of this theorem, one concludes: <u>the carrier of a subsemigroup of $\tilde{S}$ is a subsemigroup of S</u>.

<u>4.3. Kernels</u>. The next theorem, which was first formulated by Glicksberg [11, Th.3.1] for the abelian case, will be useful in our analysis. Glicksberg's proof of this depends on the group structure of the kernel; since this is no longer available to us for the non-abelian case, we use the semigroup structure of the kernel.

<u>Theorem 13</u>. Let Σ be a closed subsemigroup of $\tilde{S}$, and let S_o = carrier Σ. Then the kernel K of S_o is the carrier of the kernel $\mathcal{K}$ of Σ.

<u>Proof</u>: We show, first, that $K_o \subset$ carrier $\mathcal{K}$. This is a consequence of the fact that the carrier $\mathcal{K}$ is an ideal of S_o. For,

$$(\cup\{\text{carrier } \sigma : \sigma \in \Sigma\})(\cup\{\text{carrier } \mathfrak{k} : \mathfrak{k} \in \mathcal{K}\})$$
$$= \cup\{\text{carrier } \sigma\mathfrak{k} : (\sigma,\mathfrak{k}) \in \Sigma \times \mathcal{K}\} \subset \text{carrier } \mathcal{K}$$

and the continuity of semigroup multiplication implies

$$(\cup\{\text{carrier } \sigma\colon \sigma \in \Sigma\})^{-}(\cup\{\text{carrier } \mathfrak{K}\colon \mathfrak{K} \in \mathcal{K}\})^{-} \subset$$
$$\text{carrier } \mathcal{K};$$

that is

$$S_o(\text{carrier } \mathcal{K}) \subset \text{carrier } \mathcal{K},$$

similarly, we have

$$(\text{carrier } \mathcal{K})\ S_o \subset \text{carrier } \mathcal{K}.$$

This proves our first assertion.

To prove the other inclusion, let $\mathcal{K}_o = \{\mu \in \Sigma\colon (\text{carrier } \mu) \cap K \neq \square\}$. Then by virtue of Theorem 12, one sees that $\mathcal{K}_o$ is an ideal of Σ; since $\mathcal{K}$ is the minimal such, we must have $\mathcal{K} \subset \mathcal{K}_o$. The compactness of Σ implies that $\mathcal{K}$ is compact and hence it has idempotents (Corollary to Theorem 2, Chap. I). Let η be an idempotent measure in $\mathcal{K}$. We then assert that carrier $\eta \subset K$. For, from above, $(\text{carrier } \eta) \cap K \neq \square$ is easily seen to be an ideal of the simple semigroup carrier η [6], so we must have $(\text{carrier } \eta) \cap K = \text{carrier } \eta$. Thus, carrier $\eta \subset K$ for all idempotents η in $\mathcal{K}$. Representing $\mathcal{K} = \cup\{\eta\Sigma\eta\colon \eta^2=\eta \text{ in } \mathcal{K}\}$ (Theorem 2, Chap. I) and from

$$\text{carrier } (\eta\sigma\eta) = (\text{carrier } \eta)(\text{carrier } \sigma)(\text{carrier } \eta)$$
$$\subset K\ (\text{carrier } \sigma)K \subset K$$

for all σ in Σ. We have

$$\text{carrier } \mathcal{K} = (\cup\{\text{carrier}(\eta\sigma\eta)\colon \eta^2=\eta \in \mathcal{K},\ \sigma \in \Sigma\})^{-} \subset K$$

since K is closed.

The following is an immediate corollary of Theorem 13.

Corollary 1. If Σ is a simple subsemigroup of $\tilde{S}$, then carrier Σ is a simple subsemigroup of S.

Note that the closedness of Σ is not required. For, carrier $\Sigma^- =$ carrier Σ and the simplicity of Σ implies that Σ^- is a simple subsemigroup of $\tilde{S}$.

Corollary 2[6]. If η is an idempotent measure in $\tilde{S}$, then carrier η is a simple subsemigroup of S.

We remark that the converse of Corollary 2 need not be true; that is, for a compact simple subsemigroup of S, there need not exist an idempotent measure that carries on it, as the next example shows. The failure of the converse does not seem to have been noticed. However, the converse of Corollary 1 is true.

Example.[4] Let S be the set of ordinals less than or equal to the first uncountable ordinal. Then S, endowed with the order topology, is a compact Hausdorff space. Under this topology and the right zero multiplication, that is $xy = y$ for any x, y in S, S forms a compact simple semigroup. In the space S, the carrier of

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[4] I am indebted to Professor Frank T. Birtel for this example.

each measure in $\tilde{S}$ is a countable subset of S. So, in particular, there is no $\eta^2=\eta$ in $\tilde{S}$ with carrier $\eta = S$.

It seems plausible to conjecture that the carrier of the minimal left (right) ideal of a compact subsemigroup Σ of $\tilde{S}$ is a minimal left (right) ideal of the carrier of Σ. This, however, is erroneous. An example is given below.

<u>Example</u>. Let S be any compact Hausdorff space with Card $S \geq 2$. Then S, under the multiplication $xy = x$, forms a compact semigroup. Let η be any measure in $\tilde{S}$ with carrier $\eta = S_1$ such that Card $S_1 \geq 2$. Then $\Sigma = \{\eta\}$ is a compact subsemigroup of $\tilde{S}$ having Σ itself as its minimal left ideal; but carrier $\Sigma = S_1$ is not its own minimal left ideal. For, each element in S_1 is itself a minimal left ideal of S_1.

<u>Theorem 14</u>. Let $\mu \in \tilde{S}$, then $(1/N)\Sigma_{n=1}^{N}\mu^n$ converges, in the weak-star topology, to an idempotent measure ν carried on the kernel K_μ of the closed subsemigroup of S generated by carrier μ. If, in particular, K_μ has a unit, then carrier ν is a compact group and ν is the normalized Haar measure on K_μ.

<u>Proof</u>: Let us write $\tilde{S}$ for the unit ball $\{\mu: \mu \in M(S), \|\mu\| \leq 1\}$ of M(S), which by Theorems 7 and 8 is

compact in the weak-star topology. Let $\nu_N = (1/N)\Sigma_{n=1}^{N}\mu^n$, then since the sequence $\{\nu_N: N = 1,2,\ldots\}$ is in the compact set $\tilde{\tilde{S}}$, it must cluster somewhere in $\tilde{\tilde{S}}$. In the following, if A is a subset of $\tilde{S}$ <u>let us agree that</u> $\overline{\text{cov}}$ A <u>is the weak-star closed convex hull of</u> A. It is quite easy to see that if ν is a weak-star cluster point of $\{\nu_N: N = 1,2,\ldots\}$ then ν must be in $\overline{\text{cov}}\,\Gamma(\mu)$. From the fact that $\mu\nu_N - \nu_N = (1/N)(\mu^{n+1} - \mu)$ converges, in the weak-star topology, to 0, we have $\mu\nu = \nu$ and similarly $\nu\mu = \nu$. It follows from the continuity of the convolution that $\Gamma(\mu)\nu = \nu = \nu\Gamma(\mu)$ and furthermore ν is the zero in $\overline{\text{cov}}\,\Gamma(\mu)$. Since one can not have more than one zero in a set, and $\overline{\text{cov}}\,\Gamma(\mu)$ is compact we have $\nu = \lim_{N\to\infty} (1/N)$ $\Sigma_{n=1}^{N}\mu^n$ in the weak-star topology and $\nu^2 = \nu$,

The closed subsemigroup of S generated by the set carrier μ is

$$[\bigcup_{n=1}^{\infty} (\text{carrier } \mu)^n]^- = [\bigcup_{n=1}^{\infty} (\text{carrier } \mu^n)]^- = \text{carrier } \Gamma(\mu)$$

$$= \text{carrier } \overline{\text{cov}}\,\Gamma(\mu).$$

Now ν, the zero of the semigroup $\overline{\text{cov}}\,\Gamma(\mu)$, is the kernel of $\overline{\text{cov}}\,\Gamma(\mu)$. Therefore, by Theorem 13, carrier $\nu = K_\mu$. is the kernel of the closed semigroup carrier $\overline{\text{cov}}\,\Gamma(\mu)$ generated by carrier μ. This proves our assertion.

If, in particular, K_μ has a unit; it is a compact simple unital semigroup. Such a semigroup is necessarily a group; and hence by Wendel [31] ν is its Haar measure.

4.4. Maximal Subgroups and Normalizers. Let η be an idempotent in $\tilde{S}$ such that the carrier of η has a unit, say e. Then the carrier of the maximal subgroup $H(\eta)$ of $\tilde{S}$ is a subgroup of S having e as a unit (Theorem 15). Since $H(\eta)$ is the maximal subgroup of $\tilde{S}$ with η as unit, one might plausibly ask if carrier $H(\eta)$ is the maximal subgroup of S with e as unit. That is, is carrier $H(\eta) = H(e)$? As we shall see, in Theorem 16, this is true for a certain class of S, including the abelian ones, but fails to hold in general. In fact, carrier $H(\eta)$ is the normalizer of carrier η in $H(e)$.

Theorem 15. Let $\eta^2=\eta$ in $\tilde{S}$ be such that the carrier η has a unit e. Then the carrier of $H(\eta)$ in $\tilde{S}$ is a subgroup of S with e as unit.

Proof: It follows from Corollary 1 to Theorem 13 that the carrier of $H(\eta)$ is a simple semigroup. We show, moreover, that carrier $H(\eta)$ has e as unit and consequently carrier $H(\eta)$ is a group having e as unit. From $H(\eta) = \eta H(\eta)$ and Theorem 12, we have

$$\bigcup_{\mu\in H(\eta)} \text{carrier } \mu = \bigcup_{\mu\in H(\eta)} (\text{carrier } \eta)\cdot(\text{carrier } \mu)$$

$$= (\text{carrier } \eta)\cdot(\bigcup_{\mu\in H(\eta)} \text{carrier } \mu)$$

and hence, by the continuity of semigroup multiplication,

$$\begin{aligned}\text{carrier } H(\eta) &= \Big(\bigcup_{\mu\in H(\eta)} \text{carrier } \mu\Big)^- \\ &= (\text{carrier } \eta)\cdot\Big(\bigcup_{\mu\in H(\eta)} \text{carrier } \mu\Big)^- \\ &= \text{carrier } \eta \cdot \text{carrier } H(\eta).\end{aligned}$$

Similarly, we have

$$\text{carrier } H(\eta)\circ\text{carrier } \eta = \text{carrier } H(\eta).$$

Thus,

$$e\cdot\text{carrier } H(\eta) = \text{carrier } H(\eta) = \text{carrier } H(\eta)\cdot e$$

as was asserted.

<u>Theorem 16</u>. Let S be a compact semigroup. Let η be an idempotent in $\tilde{S}$ such that carrier η has a unit e. Then the carrier of $H(\eta)$ is the normalizer N(carrier η: H(e)) of the carrier η in H(e); in particular, if S is normal then carrier $H(\eta) = H(e)$. Conversely, if g is a closed subgroup of S and if G is the maximal subgroup of S containing g, then the normalizer N(g: G) of g in G is the carrier of some maximal subgroup of $\tilde{S}$.

<u>Proof</u>: We imbed $H(\eta)$ naturally into $H(e)^\sim$, the convolution semigroup of measures on the group H(e). This is possible, because carrier $H(\eta) \subset H(e)$ by Theorem 15, and thus the relativizations of $H(\eta)$ on H(e) gives the natural imbedding. Now viewing $H(\eta)$ as a subset of

$H(e)^{\sim}$, one finds that $H(\eta)$ is the maximal subgroup of $H(e)^{\sim}$ having $\eta|H(e)$ as unit. From Glicksberg [11, Theorem 2.3], one recognizes that carrier η is a normal subgroup of the group carrier $H(\eta)$. This proves

$$\text{carrier } H(\eta) \subset N(\text{carrier } \eta\colon H(e)).$$

For simplicity in notation, throughout the rest of the proof of this theorem, let us write N for N(carrier η: H(e)), G for carrier $H(\eta)$ and g for carrier η. Then clearly N is the maximal subgroup of H(e) in which g is a normal subgroup. Using a technique of Glicksberg [11, pp.55-57], let $T_\eta\colon (N/g)^{\sim} \longrightarrow \tilde{S}$ be the operator defined by

$$T_\eta\nu(f) = \int_{N/g}\int_g f(xy)\eta(dx)\nu(dy'),$$

for all f in C(S) and all ν in $(N/g)^{\sim}$, where y' denotes the image in N/g of y in N under the quotient homomorphism. Glicksberg [11, <u>ibid</u>.] proves that T_η takes N/g[5] topologically and isomorphically into $\tilde{S}$ and that the carrier of $T_\eta(N/g)$ is the group N. Similarly, T_η takes G/g topologically and isomorphically onto $H(\eta)$ of $\tilde{S}$. Now, as we have shown earlier, $G \subset N$ so $T_\eta(N/g) \supset T_\eta(N/g) = H(\eta)$. $T_\eta(N/g)$ is a subgroup of $\tilde{S}$ and $H(\eta)$ is the maximal such

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[5]Here, the element y' in N/g and the measure $m_{y'}$ in $N(N/g)^{\sim}$ such that carrier $m_{y'} = y'$ are identified.

subgroup. Thus $T_\eta(N/g)$ and $H(\eta)$ must coinside and hence

$$\text{carrier } H(\eta) = \text{carrier } T_\eta(N/g) = N.$$

In particular, if S is normal then carrier $H(\eta) = N(g,H(e)) = H(e)$.

Conversely, if g is a closed subgroup of S, then since the Haar measure η in $\tilde{S}$ carring on g is an idempotent measure and so carrier $H(\eta) = N(g: G)$ where G is the maximal subgroup of S containing g.

<u>4.5. Remark</u>. We conclude this chapter with the following theorem which enables us to extend all the results of Glicksberg [11] on the abelian semigroups to the normal semigroups, which contains the abelian ones as well as all not necessarily abelian groups as a proper subclass.

<u>Theorem 17</u>. $\tilde{S}$ is normal if and only if S is normal.

<u>Proof</u>: For each x in S, let m_x be the <u>point mass</u> concentrated at x; that is, if E is a Borel set in S then $m_x(E) = 1$ if and only if x is in E, and $m_x(E) = 0$, otherwise. The fundamental tool in our proof is the following consequence of Arens-Kelley [2, 3.1 and 3.2]: the mapping which takes x of S into m_x of $\tilde{S}$ is a homeomorphism

of S into $\tilde{S}$, and furthermore, $\tilde{S}$ is the weak-star closed convex hull of the set $\{m_x : x \in S\}$ of point masses. Thus, we find it convenient to use the symbol x for both the element in S and the point mass concentrated at x.

Let S be a normal semigroup, and let x be in S. Then for any sequence $\{s_1, s_2, \ldots, s_n\}$ in S, there is a sequence $\{t_1, t_2, \ldots, t_n\}$ in S such that $xs_n = t_n x$ for $n = 1, 2, \ldots, N$. Therefore, regarding x, s_n and t_n as in $\tilde{S}$, we have

$$x(\Sigma_{n=1}^{N} \alpha_n s_n) = (\Sigma_{n=1}^{N} \alpha_n t_n)x$$

for any $\alpha_n \geq 0$ with $\Sigma\alpha_n = 1$. This shows

$$x(\text{cov } S) \subset (\text{cov } S)x \subset (\overline{\text{cov}}\ S)x = \tilde{S}\, x$$

and hence $x\tilde{S} \subset \tilde{S}x$. Combining this with the dual argument we find $x\tilde{S} = \tilde{S}x$ for every point mass x. Now using the Arens-Kelley result again, and the linearity and continuity one obtains $\mu\tilde{S} = \tilde{S}\mu$ for each μ in $\tilde{S}$.

Conversely, let $\tilde{S}$ be such that $\mu\tilde{S} = \tilde{S}\mu$ for all μ in $\tilde{S}$. Then since $\mu_x\mu_y = \mu_{xy}$ one obtains easily

$$xS = Sx \qquad \text{for all } x \text{ in } S.$$

It should be remarked that the fact that allows us to substitute "normal" for "abelian" in Glicksberg's

paper [11] is: (i) S is normal if and only if $\tilde{S}$ is normal, (ii) the kernel of a normal S is a topological group.

CHAPTER V

A GENERALIZATION

5.1. Introduction. In the previous chapter, for a compact semigroup S, $\tilde{S}$ was defined to be the collection of all normalized non-negative regular Borel measures on S which under convolution and the induced weak-star topology formed a compact topological semigroup. Professor A. D. Wallace asked if the process of forming the convolution semigroup of measures might be generalized to a more general class of set functions, the so called "modular functions." The purpose of the present chapter is to settle this question in the affirmative under a slight restriction. Before we are able to state the Wallace problem more precisely, some preliminaries are necessary.

5.2. Preliminaries. Let S again in this chapter be always a compact semigroup, and let $\mathfrak{F}$ be the family of all closed subsets of S. By a modular function m on S is is meant a real-valued set function m defined on $\mathfrak{F}$ such that

$$m(A \cup B) + m(A \cap B) = m(A) + m(B),$$

for any A and B in $\mathfrak{F}$. A modular function is said to be normalized if $m(S) = 1$ and $m(\square) = 0$.

Definition 9. A modular function m on S is regular if and only if, for any F in $\mathfrak{F}$ and any $\epsilon > 0$, there is an open subset V of S containing F such that $0 \leq m(E) - m(F) < \epsilon$ for any E in $\mathfrak{F}$ such that $F \subset E \subset V^-$.

Let us write $S^\#$ for the family of all normalized regular modular functions on S. We are now in position to restate Wallace's problem more clearly: is it possible to define a "convolution" and a topology for $S^\#$ in such a way that $S^\#$ becomes a compact semigroup?

We are able to answer this for the 'isotonic' modular functions. A modular function is isotone if and only if, $A \supset B$ in $\mathfrak{F}$ implies that $m(A) \geq m(B)$.

Let us agree henceforth that $S^\#$ is the collection of all normalized regular isotonic modular functions on S. An example of such a function is a normalized regular Borel measure on S restricted to $\mathfrak{F}$. It is rather peculiar that the modular functions obtained in this way turn out to be all of $S^\#$; and furthermore there is a unique extension of an element of $S^\#$ to an element of $\tilde{S}$.

We need some additonal symbolism. Let $\mathcal{B}$ denote the family of all Borel sets in S, and let $\mathcal{V}$ be the collection of all open sets in S. We write

(5.a) $$m_o(v) = 1 - M(S\backslash v),$$

for each m in $S^{\#}$ and for all v in υ. We then define a transformation ^{t} on $S^{\#}$ by

(5.b) $$m^t(B) = \inf \{m_o(V) : B \subset V \in \upsilon\}$$

for each m in $S^{\#}$ and for all Borel set B in $\mathcal{B}$.

5.3. A Convolution Semigroup of Modular Functions.

Theorem 18. The transformation ^{t} takes $S^{\#}$ in one-to-one fashion onto $\tilde{S}$. Indeed, we have the following relations:

$$m^t|\mathfrak{F} = m \quad \text{and} \quad (\mu|\mathfrak{F})^t = \mu$$

for every m in $S^{\#}$ and for every μ in $\tilde{S}$.

We divide the proof of this theorem into the following steps.

Lemma 5. If U, V are in $\mathcal{V}$ and if F is in $\mathfrak{F}$ such that $U \subset F \subset V$. Then

(5.c) $$m_o(U) \leq m(F) \leq m_o(V)$$

for any m in $S^{\#}$.

Proof: This is straightforward from (5.a).

Lemma 6. The modular function m_o is normalized, isotonic, countably additive on $\mathcal{V}$ for each m in $S^{\#}$.

<u>Proof</u>: It is clear that $m_o(\square) = 0$, $m_o(S) = 1$ and that $U = U^o \supset V^o = V$ implies $m_o(U) \geq m_o(V)$. Furthermore, m_o is finitely additive; for if U and V are any disjoint open sets, then

$$\begin{aligned} m_o(U \cup V) &= 1 - m((S\setminus U) \cap (S\setminus V)) \\ &= 1 - [m(S\setminus U) + m(S\setminus V) - m((S\setminus U) \cup (S\setminus V))] \\ &= [1 - m(S\setminus U)] + [1 - m(S\setminus V)] \\ &= m_o(U) + m_o(V), \end{aligned} \tag{5.d}$$

for $S = (S\setminus U) \cup (S\setminus V)$ and m is a normalized modular function.

Now we show that for any sequence $\{V_i : i \geq 1\}$ of open sets V_i,

$$\Sigma_{i\geq 1} m_o(V_i) \geq m_o(\cup_{i\geq 1} V_i). \tag{5.e}$$

For any $\epsilon > 0$, since $S\setminus(\cup_{i\geq 1} V_i) = \cap_{i\geq 1}(S\setminus V_i) \in \mathfrak{F}$ and by regularity of m, there is an open set W containing $\cap_{i\geq 1}(S\setminus V_i)$ such that

$$m(\cap_{i\geq 1}(S\setminus V_i)) + \epsilon \geq m(W^-). \tag{5.f}$$

Compactness of S, then, yields a positive integer n such that

$$W \supset \bigcap_{n\geq i\geq 1} (S\setminus V_i). \tag{5.g}$$

It follows then from (5.a), (5.c), (5.d), (5.f), and (5.g) that

$$\sum_{m\geq i\geq 1} m_o(V_i) + \epsilon \geq m_o(\bigcup_{i\geq 1} V_i)$$

and hence

$$\Sigma_{i\geq 1} m_o(V_i) + \epsilon \geq m_o(\cup_{i\geq 1} V_i).$$

Since ϵ was arbitrary, (5.e) is thus proved.

Finally, with an additional assumption that $\{V_i : i \geq 1\}$ is disjoint, we have to show that

(5.h) $$\Sigma_{i\geq 1} m_o(V_i) \leq m_o(\cup_{i\geq 1} V_i).$$

From (5.c) and (5.d), we have

$$\Sigma_{n\geq i\geq 1} m_o(V_i) = m_o(\cup_{n\geq i\geq 1} V_i) \leq m(\cup_{i\geq 1} V_i)$$

for every positive integer n. Therefore (5.h) holds, and hence the lemma is proved.

<u>Lemma 7</u>. If $m \in S^{\#}$ then m^t is a countably additive measure on $\mathcal{B}$.

<u>Proof</u>: From (5.b) we have $m^t(\square) = 0$. Let $\{B_i : i \geq 1\}$ be a sequence of Borel sets such that $\cup\{B_i : i \geq 1\}$ is also a Borel set. Then for any $\epsilon > 0$ and for each positive integer i, there is an open set $V_i \supset B_i$ such that $m_o(V_i) \leq m^t(B_i) + \frac{\epsilon}{2^i}$. Therefore, from (5.d) and (5.e), we have

$$m^t(\cup_{i\geq 1} B_i) \leq m_o(\cup_{i\geq 1} V_i) \leq \Sigma_{i\geq 1} m_o(V_i) \leq \Sigma_{i\geq 1} m^t(B_i) + \epsilon,$$

and thus,

$$(5.i) \qquad m^t(\bigcup_{i\geq 1} B_i) \leq \sum_{i\geq 1} m^t(B_i).$$

These together with the isotony of m^t show that m^t is an outer measure on the σ-algebra $\mathcal{B}$.

Since m is regular, by some standard computations, all sets in $\mathfrak{F}$ are m^t-sets (= outer measurable sets). Now a celebrated theorem of Carathéodory (see for instance [10, p. 134]), tells us that all Borel sets are m^t-sets upon which m^t is countably additive.

<u>Lemma 8</u>. If $m \in S\#$ then m^t is regular.

<u>Proof</u>: This follows from the fact that

$$m^t(F) = \inf\{m_o(V) : F \subset V \in \mathcal{V}\}$$
$$= \inf\{m^t(V) : F \subset V \in \mathcal{V}\} .$$

<u>Lemma 9</u>. If $m \in S\#$ then $m^t|\mathfrak{F} = m$.

<u>Proof</u>: Since m is regular, for any F in $\mathfrak{F}$ and any $\epsilon > 0$ there is an open set V such that

$$F \subset V \text{ and } m(V^-) \leq m(F) + \epsilon.$$

Using (5.b) and (5.c) several times we arrive at

$$m(F) \leq m^t(F) \leq m_o(V) \leq m(V^-) \leq m(F) + \epsilon$$

and see $m(F) = m^t(F)$ for all F in $\mathfrak{F}$.

<u>Lemma 10</u>. If $\mu \in \tilde{S}$ then $(\mu|\mathfrak{F})^t = \mu$.

<u>Proof</u>: It is fairly clear that $\mu|\mathfrak{F}$ belongs to $S^{\#}$. Let us denote $(\mu|\mathfrak{F})^t$ by ν. Then by Lemma 9,

$$\nu(F) = \mu(F)$$

for every closed set F, and hence

$$\nu(V) = \mu(V)$$

for each open set V. We have then, by the regularity of μ and by (5.b),

$$\nu(B) = \mu(B)$$

for every Borel set B.

The proof of Theorem 18 is now clear from Lemmas 5-10. We are now ready to state our main theorem of this chapter.

<u>Theorem 19</u>. Let S be a compact semigroup. Then the set $S^{\#}$ of all normalized regular isotonic modular functions may be introduced a convolution and a topology in such a way that $S^{\#}$ is topologically isomorphic to $\tilde{S}$.

<u>Proof</u>: We define, by virtue of Theorem 18, the convolution $*$ on $S^{\#}$ naturally by

$$m * n = (m^t \cdot n^t) \mid \mathfrak{F}$$

for all m, n in $S^{\#}$. Where $\cdot$ in the right hand side means the convolution of measures in the usual sense. Topologize $S^{\#}$ in such a way that a subset Σ is open if and only if $\Sigma^t = \{\sigma^t : \sigma \in \Sigma\}$ is open in $\tilde{S}$. Then the mapping $h: S^{\#} \longrightarrow \tilde{S}$ defined by $h(m) = m^t$ for all m in $S^{\#}$ is an isomorphism as well as a homeomorphism.

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This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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